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APPROXIMATE HOMOMORPHISMS

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Every mapping between finite Boolean algebras which is approximately a homomorphism with respect to some measure φ on the range (see Definition 2) can be approximated by a homomorphism within a constant error. An analogous statement fails in case when φ is a pathological submeasure. The key to our proof is the fact that a subset of a finite Boolean algebra $\{0,1\}^{[m]}$ which "almost everywhere" looks like an ultrafilter has to be close to some fixed ultrafilter.

This note can be thought of as belonging to the general "stability program," proposed long ago by S. Ulam ([17; VI.1], see also [18; V.4]). A typical result of this sort, due to D.H. Hyers ([6]) is: For a function $H:\mathbb{R}\to\mathbb{R}$ which is ε-approximately additive, namely such that for all a,b we have

$$|H(a+b)-H(a)-H(b)|\leq\varepsilon,$$

there is an additive function $f:\mathbb{R}\to\mathbb{R}$ such that $|f-H|\leq\varepsilon$. This theorem was later generalized to transformations of some other spaces (see [18; V.4]). In [8], N. Kalton and J. Roberts proved an analogous statement for set-mappings: If $H:\{0,1\}^{[m]}\to\mathbb{R}$ is ε-approximately additive, then there is an additive mapping $f:\{0,1\}^{[m]}\to\mathbb{R}$ such that $|f-H|\leq 45\varepsilon$. We make a further increase in the complexity by considering ε-approximate homomorphisms $H:\{0,1\}^{[m]}\to\{0,1\}^{[n]}$ with respect to some submeasure on the range. Our main result is that, assuming the submeasure is nonpathological (see below), there is always a homomorphism $\Phi:\{0,1\}^{[m]}\to\{0,1\}^{[n]}$ such that $\|\Phi\Delta H\|_\varphi\leq 521\varepsilon$. This increase in complexity leads to the occurrence of an instability in Ulam's sense. Namely, dropping the requirement that the submeasure be nonpathological results in the non-existence of universal constant K such that every ε-approximate homomorphism can be $K\varepsilon$-approximated by a homomorphism.

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Definition 1. A map $\varphi\colon\mathcal{P}(X)\to[0,\infty]$ is a *submeasure supported by* X if

$\varphi(\emptyset)=0$

$\varphi(A)\leq\varphi(A\cup B)$ (φ is *monotonic*), and

$\varphi(A\cup B)\leq\varphi(A)+\varphi(B)$ (φ is *subadditive*)

for all $A,B\subseteq X$. The *norm of* φ is defined by $\|\varphi\|=\varphi(X)$.

By s,t,u,v we always denote elements of $\{0,1\}^{[m]}$ or $\{0,1\}^{[n]}$, which are identified with subsets of $\{1,\dots,m\}$ or $\{1,\dots,n\}$, respectively. By s^{C} we denote the complement of s, which is equal to either $\{1,\dots,m\}\setminus s$ or $\{1,\dots,n\}\setminus s$, which would be clear from the context.

Definition 2. Assume $\{1,\dots,m\}$ supports a submeasure φ and fix $\varepsilon,\delta>0$. A mapping $H\colon\{0,1\}^{[m]}\to\{0,1\}^{[n]}$ is an ε*-approximate homomorphism* (with respect to φ) if for all s,t we have

$$\varphi((H(s)\cup H(t))\Delta H(s\cup t))<\varepsilon$$
$$\varphi(H(s^{\mathsf{C}})\Delta H(s)^{\mathsf{C}})<\varepsilon$$

A homomorphism $\Phi\colon\{0,1\}^{[m]}\to\{0,1\}^{[n]}$ is a δ*-approximation for* H if

$$\varphi(\Phi(s)\Delta H(s))<\delta$$

for all s.

In this note we study the question of how well can an ε-approximate homomorphism be approximated by a homomorphism. In particular

Question 3. Is there a universal constant K such that for every $\varepsilon>0$ every ε-approximate homomorphism can be $K\cdot\varepsilon$-approximated by a homomorphism?

Our own interest in this question comes from a study of a conjecture of Todorcevic ([15, Problem 1]) about simple liftings of homomorphisms between quotient algebras over analytic ideals on the integers, to which Question 3 turns out to be equivalent (see [3], [4] for more details). We shall prove that this question has a positive answer for a natural class of submeasures which includes all measures (Theorems 4 and 5), but that in general it has a negative answer (Theorem 7). A submeasure φ is *pathological* if it differs from the supremum of all measures it dominates, and the function $\hat\varphi$ defined by

$$\hat\varphi(s)=\sup_{\nu\leq\varphi}\nu(s)$$

(ν ranges over measures dominated by φ) is a maximal nonpathological submeasure dominated by φ. Let

$$P(\varphi)=\sup_{\hat\varphi(s)\neq 0}\frac{\varphi(s)}{\hat\varphi(s)}.$$

Our definition of a pathological submeasure differs from the standard one, where a submeasure is called pathological if $\hat\varphi$ is identically equal to zero. However, if the underlying set is finite (as it is in our case) there are no nonvanishing submeasures which are pathological in this stronger sense, and this is why we choose this as our definition. The notion of a pathological submeasure was first considered by V. A. Popov ([11]), and since then pathological submeasures have drawn considerable attention (see e.g. [2], [1], [16], [12]), especially in connection with the Control Measure Problem, or the Maharam Problem (see [7], [5]). Our results give a new example of a pathological submeasure and a new look at pathological and nonpathological submeasures. It is interesting to note that the Kalton–Roberts result was discovered in the course of solving a reduced version of the Control Measure Problem asked by Talagrand ([12, page 102]). We should also remark that, although this result apparently looks very similar to ours, the two proofs are rather different: the main tool used in our proof is Fubini's theorem, while the crucial combinatorial part of Kalton–Roberts proof rests on a lemma about concentrators (see [10]). It does not seem plausible that either one of these two results can be easily deduced from another.

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1. Stability

Define $K_\varphi \in \mathbb{R}^+ \cup \{\infty\}$ to be the supremum of all K such that some ε-approximate homomorphism H can not be $K\cdot\varepsilon$-approximated by a homomorphism.

Theorem 4. *There is a universal constant $K_M \leq 521$ such that $K_\mu \leq K_M$ for every measure μ on some $\{0,1\}^{[n]}$.*

Theorem 5. *This constant also satisfies $K_\varphi \leq K_M \cdot P(\varphi)$ for every submeasure φ.*

A simple compactness argument (see e.g. in [8; p. 809]) shows that Theorems 4 and 5 also remain true in the infinite case. The only place in the proof of Theorem 4 where we use the fact that μ is a measure instead of a submeasure is to get Fubini's theorem for the product of μ and another measure. This is not surprising, since by a result of Christensen ([1]), pathological submeasures (for which Theorem 4 fails) can be characterized by the failure of Fubini's theorem.

In our proof of Theorem 4 we shall need a result which says that a family of subsets of some finite set which almost everywhere looks like an ultrafilter must be close to some principal ultrafilter, $\langle k\rangle = \{s \in \{0,1\}^{[m]} : k \in s\}$. A related study of subsets of the discrete cube can be found e.g. in [13] and [14; Theorem 1.1].

Let ν denote the uniform probabilistic measure on $\{0,1\}^{[m]}$, and let ν^2 denote the product measure on $\{0,1\}^{[m]} \times \{0,1\}^{[m]}$.

Theorem 6. *If $\delta > 0$ is small enough (say, $\delta \leq 1/85$) and $A \subseteq \{0,1\}^{[m]}$ satisfies*

(A1) $\nu^2((A \times \{0,1\}^{[m]} \cup \{0,1\}^{[m]} \times A) \Delta \{\langle s,t\rangle : s \cup t \in A\}) < \delta$,

(A2) $\nu^2((A \times A) \Delta \{\langle s,t\rangle : s \cap t \in A\}) < \delta$,

(A3) $\nu\{s : s, s^{\complement} \notin A$ *or* $s, s^{\complement} \in A\} < \delta$,

then there is a $k \in \{1,\dots,m\}$ such that $\nu\{s \in A : k \in s\} > (1-28\delta)/2$. In particular $\nu(A \Delta \langle k\rangle) < 29\delta$.

Proof. For $I \subseteq \{1,\dots,m\}$ and s we define

$$s^I = (I \cap s) \cup (I^{\complement} \setminus s)$$
$$A^2(I) = \{s : s, s^I \in A\}$$
$$E_I = \nu(A^2(I)).$$

The number E_I is, in some sense, a measure of how A concentrates on I. For example, if A is an ultrafilter, then E_I is equal to either $1/2$ or 0, depending on whether I is in A or not. Note that (A3) implies

$$\text{(A)} \qquad E_\emptyset < \delta \quad \text{and} \quad E_{\{1,\dots,m\}} = \nu(A) > \frac{1}{2} - \delta.$$

The set of all $s \in \{0,1\}^{[m]}$ such that $s, s^{\complement} \notin A$ or $s^I, (s^I)^{\complement} \notin A$ is, by (A3), of measure $< 2\delta$, and for each s outside of this set we either have $s \in A^2(I) \cup A^2(I^{\complement})$ or $s^{\complement} \in A^2(I) \cup A^2(I^{\complement})$. (To see this, consider the possible cases: if $s, s^I \in A$ then $s \in A^2(I)$, if $s, (s^I)^{\complement} \in A$ then $s \in A^2(I^{\complement})$, and so on). Therefore

$$\text{(B)} \qquad E_I + E_{I^{\complement}} \geq \frac{1-2\delta}{2}.$$

We claim that

$$\text{(C)} \qquad E_I \cdot E_{I^{\complement}} < 5\delta.$$

To see this, note that the left-hand side is equal to $\nu(A^2(I)) \cdot \nu(A^2(I^{\complement})) = \nu^2(A^2(I) \times A^2(I^{\complement}))$ and that $C = A^2(I) \times A^2(I^{\complement}) = C_0 \cup C_1 \cup C_2$, where

$$C_0 = \{\langle s,t\rangle \in C : s \cap t, s^I \cap t^{I^{\complement}} \in A\}$$
$$C_1 = \{\langle s,t\rangle \in C : s \cap t \notin A\}$$
$$C_2 = \{\langle s,t\rangle \in C : s^I \cap t^{I^{\complement}} \notin A\}.$$

It suffices to show that $\nu^2(C_0) < 3\delta$, $\nu^2(C_1) < \delta$ and $\nu^2(C_2) < \delta$. The latter two inequalities follow immediately from (A2), and C_0 has the same size as the set

$$\{\langle u, v\rangle : \langle (u\cap I)\cup(v\cap I^{\mathsf{C}}), (v\cap I)\cup(u\cap I^{\mathsf{C}})\rangle \in C \quad \text{and} \quad v\cap u, v^{\mathsf{C}}\cap u \in A\}.$$

To see that the measure of this set is $<3\delta$, we split it into three pieces, depending on whether $v\notin A$, $v^{\mathsf{C}}\notin A$, or $v, v^{\mathsf{C}}\in A$. Each of these pieces has a measure less than δ (by (A2) and (A3)), and therefore $\nu^2(C_2)<3\delta$. Let $\gamma=\sqrt{(1/2-\delta)^2-20\delta}$, and note that $(1-21\delta)/2<\gamma<1/2$. By using (B) and (C), we get $E_I(1/2-E_I-\delta)<5\delta$ and therefore

(D) $$E_I < \frac{1-2\delta-2\gamma}{4} \qquad \text{or} \qquad \frac{1-2\delta+2\gamma}{4} < E_I.$$

Let $E_k=E_{\{1,\ldots,k\}}$. By (A) there exists a minimal $k\le m$ such that $E_k>(1–2\delta+2\gamma)/2$, and by (D) for this k we have

(E) $$E_k - E_{k-1} > \gamma.$$

This means that A concentrates on $\{1,\ldots,k\}$, but not on $\{1,\ldots,k-1\}$. We shall prove that the principal ultrafilter $\langle k\rangle=\{s\in\{0,1\}^{[m]}:k\in s\}$ is close to A. In order to do so, we have to prove that the set

$$G = \{s \in A^2(\{1,\ldots,k\}) : k \notin s\}$$

is small. Consider the set

$$F = \{s : s\in A \quad \text{and} \quad s\cup\{k\}\notin A\}.$$

Fix $s\in G\setminus A^2(\{1,\ldots,k-1\})$. Then both s and $s^{\{1,\ldots,k\}}$ are in A but $s^{\{1,\ldots,k-1\}}$ is not. Since $k\notin s$ (because $s\in G$), this implies $s^{\{1,\ldots,k-1\}} = s^{\{1,\ldots,k\}}\cup\{k\}$, and therefore $s^{\{1,\ldots,k\}}\in F$. So we have

(F) $$\nu(G) \le E_{k-1} + \nu(F).$$

We claim that

(G) $$\nu(F) = \nu\{s : s\in A \quad \text{and} \quad s\cup\{k\}\notin A\} < \frac{8\delta}{1-2\delta}.$$

To see this, let $L=\{t:k\notin t$ and either t or $t\cup\{k\}$ is in $A\}$. We have

$$\begin{aligned}\{t : t, t^{\mathsf{C}}\notin A\} &\supseteq \{t : k\notin t \quad \text{and} \quad t, t\cup\{k\}, t^{\mathsf{C}}\setminus\{k\}, t^{\mathsf{C}}\notin A\}\\ &\supseteq \{t : k\notin t \quad \text{and} \quad t, t^{\mathsf{C}}\setminus\{k\}\notin L\}\\ &= M,\ \text{say},\end{aligned}$$

and therefore by (A3) we have

$$\text{(H)} \qquad \nu(L) \geq \frac{\nu\{t : k \notin t\} - \nu(M)}{2} \geq \frac{1-2\delta}{4}.$$

Consider sets

$$F_0 = \{\langle s,t\rangle \in F \times L : s \cap t = s \cap (t \cup \{k\}) \notin A\}$$
$$F_1 = \{\langle s,t\rangle \in F \times L : s \cap t = s \cap (t \cup \{k\}) \in A\}.$$

For $t \in L$ define

$$\hat{t} = \begin{cases} t, & \text{if } t \in A \\ t \cup \{k\}, & \text{otherwise.} \end{cases}$$

Thus (A2) implies $\nu^2\{\langle s,\hat{t}\rangle : \langle s,t\rangle \in F_0\} < \delta$ and $\nu^2\{\langle s \cup \{k\}, t\rangle : \langle s,t\rangle \in F_1\} < \delta$. Since $s \mapsto s \cup \{k\}$ is a $1-1$ mapping on F and $t \mapsto \hat{t}$ is a $1-1$ mapping on L, this implies that $\nu^2(F \times L) = \nu^2(F_0) + \nu^2(F_1) < 2\delta$. Therefore (H) implies that $\nu(F) < 8\delta/(1-2\delta)$, as required. Finally we have

$$\begin{aligned}
\nu\{s \in A : k \in s\} &\geq \nu\{s \in A^2(\{1,\dots,k\}) : k \in s\} \\
&= E_k - \nu\{s \in A^2(\{1,\dots,k\}) : k \notin s\} \\
&> E_k - E_{k-1} - \frac{8\delta}{1-2\delta} \qquad \text{(by (F) and (G))} \\
&> \gamma - \frac{8\delta}{1-2\delta} \qquad \text{(by (E)).}
\end{aligned}$$

Therefore

$$\begin{aligned}
\nu(A^2(\{k\})) &\geq \nu\{s : k \in s \quad \text{and} \quad s, s^{\complement} \cup \{k\} \in A\} \\
&\geq 2\nu\{s \in A : k \in s\} - \frac{1}{2} \\
&> 2\left(\gamma - \frac{8\delta}{1-2\delta}\right) - \frac{1}{2}.
\end{aligned}$$

Since $\delta \leq 1/85$ and (G) imply $\nu(F) < 33\delta/4$ and $\gamma > (1-21\delta)/2$, we have $10\gamma > 3-64\delta$, and therefore

$$\nu(A^2(\{k\}) > 2\left(\gamma - \frac{33\delta}{4}\right) - \frac{1}{2} > \frac{1-2\gamma-2\delta}{4}.$$

this formula and (D) together imply $\nu(A^2(\{k\})) = E_{\{k\}} > (1-2\delta+2\gamma)/4$ and

$$\nu\{s \in A : k \in s\} > \nu(A^2(\{k\}) - \nu\{s \in A : s \cup \{k\} \notin A\} > \frac{1-28\delta}{2}.$$

Also, $\nu(A \Delta \langle k\rangle) = \nu(A) + \nu(\langle k\rangle) - 2\nu(A \cap \langle k\rangle) \leq 1/2 + \delta/2 + 1/2 - (1-28\delta) < 29\delta$. ∎

Remark. Note that the proof of 6 shows that (under the same assumptions and using the same notation) for all $I \ni k$ we have

$$\nu\{s : s, s^I \in A\} > \frac{1-2\delta+2\gamma}{4} > \frac{2-23\delta}{4}. \tag{I}$$

Proof. (Theorem 4) Assume $H\colon\{0,1\}^{[m]} \to \{0,1\}^{[n]}$ is an ε-approximate homomorphism with respect to a measure μ on $\{0,1\}^{[n]}$. For $j \le n$ let

$$A_j = \{s \in \{0,1\}^{[m]} : H(s) \ni j\}.$$

Let $\delta = 1/85$, and for $i = 1,2,3$ let N_i be the set of all $j \le n$ for which (Ai) fails for the set A_j. Since for all $s,t \in \{0,1\}^{[m]}$ we have $\mu((H(s)\cup H(t))\Delta H(s\cup t)) < \varepsilon$, Fubini's theorem applied to the product $(\{0,1\}^{[m]})^2 \times [n]$ implies $\mu(N_1) < \varepsilon/\delta$. To get a bound for $\mu(N_2)$, note that

$$\begin{aligned}\varphi((H(s)\cap H(t))\Delta H(s\cap t)) &= \varphi((H(s)\cap H(t))^{\mathsf{C}}\Delta H(s\cap t)^{\mathsf{C}})\\ &\le \varphi((H(s)^{\mathsf{C}}\cup H(t)^{\mathsf{C}})\Delta(H(s^{\mathsf{C}})\cup H(t^{\mathsf{C}}))\\ &+ \varphi((H(s^{\mathsf{C}})\cup H(t^{\mathsf{C}}))\Delta H(s^{\mathsf{C}}\cup t^{\mathsf{C}}))\\ &+ \varphi(H((s\cap t)^{\mathsf{C}})\Delta H(s\cap t)^{\mathsf{C}})\\ &\le 4\varepsilon.\end{aligned}$$

By Fubini's theorem we have $\mu(N_2) < 4\varepsilon/\delta$, and similarly, $\mu(N_3) < \varepsilon/\delta$. Therefore

$$\mu\Big(\bigcup_{i=1}^{3} N_i\Big) < \frac{6\varepsilon}{\delta}. \tag{J}$$

For every $j \in \{1,\dots,n\} \setminus \bigcup_{i=1}^{3} N_i$ the set A_j satisfies assumptions of Theorem 6, so for such a j let $k(j) \le m$ be such that $\nu(A_j\Delta\langle k(j)\rangle) < 29\delta$. Assume for a moment that $N_i = \emptyset$ for all i, and define a homomorphism $\Phi\colon\{0,1\}^{[m]} \to \{0,1\}^{[n]}$ by

$$\Phi(s) = \{j : k(j) \in s\}$$

Claim. The homomorphism Φ is a $(16/(2-23\delta)+1)\varepsilon$-approximation to H.
Proof. Assume it is not. Thus for some $I \subseteq \{1,\dots,m\}$ we have

$$\mu(\Phi(I)\Delta H(I)) > \left(\frac{16}{2-23\delta}+1\right)\varepsilon.$$

Since $(H(I)\setminus\Phi(I))\Delta(\Phi(I^{\mathsf{C}})\setminus H(I^{\mathsf{C}})) \subseteq H(I)\Delta H(I^{\mathsf{C}})^{\mathsf{C}}$ and the measure of the latter set is not bigger than ε, we can assume (possibly by replacing I with I^{C}) that

$$\mu(\Phi(I)\setminus H(I)) > \frac{8\varepsilon}{2-23\delta}.$$

Using the notation defined in the proof of Theorem 6, by (I) we have

$$\nu\{s : s, s^I \in A_j\} > \frac{2-23\delta}{4}$$

for all $j \in \Phi(I) \setminus H(I)$. By Fubini's theorem, there are a $C_1 \subseteq \Phi(I) \setminus H(I)$ and a $\bar{u} \in \{0,1\}^{[m]}$ such that

$$\mu(C_1) \geq \frac{2-23\delta}{4} \cdot \frac{8\varepsilon}{2-23\delta} = 2\varepsilon$$

and $\bar{u}, \bar{u}^I \in \bigcap_{j\in C_1} A_j$. Finally we have

$$0 = \mu(C_1 \cap H(I)) > \mu(C_1 \cap (H(\bar{u}) \cap H(\bar{u}^I))) - 2\varepsilon = \mu(C_1) - 2\varepsilon \geq 0,$$

a contradiction. ∎

If the set $\bigcup_{i=1}^3 N_i$ is not empty, then extend the function k obtained using Theorem 6 by letting $k(j)=1$ for all j in this set, and define Φ as in Claim above. Then proof of the Claim shows that

$$\nu((\Phi(s)\Delta H(s) \cap (\{0,1\}^{[n]} \setminus \bigcup_{i=1}^{3} N_i)) \leq \left(\frac{16}{2-23\delta}+1\right)\varepsilon.$$

Since $16/(2-23\delta)+1+6/\delta \leq 521$ for $\delta \leq 1/85$, (J) implies that Φ is a $521\cdot\varepsilon$-approximation to H. ∎

The definitions of A_j's and Φ did not depend on either μ or ε, and therefore we have proved the following stronger version of Theorem 4:

Theorem 4*. *For every map $H\colon\{0,1\}^{[m]} \to \{0,1\}^{[n]}$ there is a homomorphism $\Phi\colon\{0,1\}^{[m]} \to \{0,1\}^{[n]}$ such that for every measure μ supported by $\{1,\dots,n\}$ and every ε, if H is an ε-approximate homomorphism with respect to μ, then Φ is an $\varepsilon\cdot K_M$-approximation to H.*

∎

Proof. (Theorem 5) Let $H\colon\{0,1\}^{[m]} \to \{0,1\}^{[n]}$ be an ε-approximate (with respect to φ) homomorphism, and let $\Phi\colon\{0,1\}^{[m]} \to \{0,1\}^{[n]}$ be as guaranteed by Theorem 4*. To prove that Φ is as required, fix $s \in \{0,1\}^{[m]}$, let $u = H(s)\Delta\Phi(s)$ and find a measure $\mu \leq \varphi$ on $\{0,1\}^{[n]}$ such that $P(\varphi)\cdot\mu(u) = \varphi(u)$. Such a μ exists because $\{0,1\}^{[n]}$ is finite, and we have sup = max. Since $\mu \leq \varphi$ and H is an ε-approximate homomorphism with respect to φ, it is an ε-approximate homomorphism with respect to μ as well. Therefore Theorem 4* implies $\varphi(u) = P(\varphi)\cdot\mu(u) \leq \varepsilon\cdot K_\varphi$. ∎

An alternative proof of Theorem 5 can be obtained by using Christensen's characterization of nonpathological submeasures as those for which Fubini's theorem holds (see [1]).

2. A pathological submeasure

In this section we show that Question 3 has a negative answer in general.

Theorem 7. *For every $M<\infty$ there is a submeasure φ such that $K_\varphi>M$.*

Proof. For $m>2^{3M+2}$ let $n=2^{2^m}$, so we can identify $\{1,\dots,n\}$ with the set N of all subsets X of $\{0,1\}^{[m]}$. We shall denote elements of N by X,Y,Z. Define $H:\{0,1\}^{[m]}\to\{0,1\}^{[n]}$ by

$$H(s)=B_s=\{X\in N: s\in X\}.$$

For $k=1,\dots,m$ let $\langle k\rangle$ denote the principal ultrafilter $\{s\in\{0,1\}^{[m]}:k\in s\}$. Define the submeasure $\varphi=\varphi_m$ supported by N by letting

$$\varphi\{\langle 1\rangle,\langle 2\rangle,\dots,\langle m\rangle\}=0$$

and for $\mathcal{A}$ disjoint from $\{\langle 1\rangle,\dots,\langle m\rangle\}$ let

$$\begin{aligned}\varphi(\mathcal{A})=\min\{|X|: X\in N &\text{ is such that}\\ &Y\cap X\neq\langle k\rangle\cap X \text{ for all } Y\in\mathcal{A} \text{ and } k=1,\dots,m\}.\end{aligned}$$

In other words, if for $X\in N$ we define

$$\mathcal{C}_X=\{Y\in N:(Y\Delta\langle k\rangle)\cap X\neq\emptyset \text{ for all } k=1,\dots,m\},$$

then $\varphi(\mathcal{A})$ is equal to the smallest size of $X\in N$ such that $\mathcal{A}\subseteq\mathcal{C}_X$.

Claim 1. The function φ is a submeasure.

Proof. The monotonicity of φ is trivial, while its subadditivity follows from the formula $\mathcal{C}_X\cup\mathcal{C}_Y\subseteq\mathcal{C}_{X\cup Y}$. ∎

Claim 2. The function H is a $(3+\varepsilon)$-approximate homomorphism with respect to φ, for every $\varepsilon>0$.

Proof. Note that for every $s\in\{0,1\}^{[m]}$ we have $H(s)^{\complement}\Delta H(s^{\complement})\subseteq\mathcal{C}_{\{s,s^{\complement}\}}$, therefore $\varphi(H(s)^{\complement}\Delta H(s^{\complement}))\leq 2$. Similarly for all s,t we have $(H(s)\cup H(t))\Delta H(s\cup t)\subseteq\mathcal{C}_{\{s,t,s\cup t\}}$, thus $\varphi((H(s)\cup H(t))\Delta H(s\cup t))\leq 3$. ∎

If $K_\varphi\leq M$ then H can be $3M$-approximated by a homomorphism $\Phi:\{0,1\}^{[m]}\to\{0,1\}^{[n]}$. For $s\in\{0,1\}^{[m]}$ let $\mathcal{A}_s=\Phi(s)\Delta H(s)$, let $Y_s\subseteq\{0,1\}^{[m]}$ be of size $3M$ such that $\mathcal{A}_s\subseteq\mathcal{C}_{Y_s}$, and let $X_s=Y_s\cup\{s\}$. Then for $k=1,\dots,m$ we have

$$\Phi(\{k\})\supseteq H(\{k\})\setminus\mathcal{A}_{\{k\}}.$$

Since any $Y\in H(\{k\})$ satisfying $(Y\Delta\langle k\rangle)\cap X_{\{k\}}=\emptyset$ is not in $\mathcal{A}_{\{k\}}$, we have

$$\begin{aligned}|\Phi(\{k\})|&\geq|\{Y\in H(\{k\}):(Y\Delta\langle k\rangle)\cap X_{\{x\}}=\emptyset\}|\\ &=|\{Y\in N:\{k\}\in Y \quad\text{and}\quad (Y\Delta\langle k\rangle)\cap X_{\{x\}}=\emptyset\}|\geq 2^{2^m-3M-2}.\end{aligned}$$

The sets $\Phi(\{k\})$ $(k=1,\ldots,m)$ are pairwise disjoint, so we have

$$n \geq |\bigcup_{k=1}^{m} \Phi(\{k\})| \geq m \cdot 2^{2^m - 3M - 2} > 2^{2^m} = n,$$

a contradiction. Therefore $K_\varphi > M$, as promised. ∎

Theorems 4 and 7 together show that the submeasure φ_m defined during the course of proving Theorem 7 is pathological; more precisely, that

$$P(\varphi_m) > \frac{M}{K_{\varphi_m}} \geq \frac{\log_2(m) - 2}{1563}.$$

We shall now give a direct argument to show that $P(\varphi_m)$ is much bigger.

Theorem 8. *If $\varphi = \varphi_m$ is as in the proof of Theorem 7, then $P(\varphi) > 2^{2m-5}$.*

Proof. Let k, l be integers such that $kl \leq 2^{m-1}$, and let $s_1, s_2, \ldots, s_{kl}$ be a $1-1$ sequence of elements from $\{s \in \{0,1\}^{[m]} : 1 \in s\}$ (all we need is that all s_i are pairwise distinct and that $s_i \neq s_j^{\complement}$ for all i,j). Let

$$C_i = \mathcal{C}_{\{s_i, s_i^{\complement}\}} = \{X \in N : s_i, s_i^{\complement} \in X \text{ or } s_i, s_i^{\complement} \notin X\}, \qquad \text{for } i \leq kl.$$

We shall restrict φ to the set

$$\mathcal{U} = \{X \in N : |\{i : X \in C_i\}| = l\}$$

and normalize it by

$$\varphi'(\mathcal{A}) = \frac{\varphi(\mathcal{A})}{2(k-1)l + 2}, \qquad \text{for } \mathcal{A} \subseteq \mathcal{U}.$$

Note that $P(\varphi) \geq P(\varphi')$, hence we can concentrate on φ'.

Claim 1. $\varphi'(\mathcal{U}) = 1$.

Proof. To see that $\varphi'(\mathcal{U}) \leq 1$, let $Y = \{s_i, s_i^{\complement} : i \leq (k-1)l + 1\}$. Since for every $X \in \mathcal{U}$ there is an $i \leq (k-1)l + 1$ such that $X \in C_i \subseteq \mathcal{C}_Y$, we have $\mathcal{U} \subseteq \mathcal{C}_Y$ and $\varphi'(\mathcal{U}) \leq |Y|/(2(k-1)l + 2) = 1$. To see that $\varphi'(\mathcal{U}) \geq 1$, fix some $\mathcal{A} \subseteq \mathcal{U}$ such that $\varphi'(\mathcal{A}) < 1$. We can assume that $\mathcal{A} = \mathcal{C}_Y$ for some $Y \subseteq \{0,1\}^{[m]}$ of size at most $2(k-1)l + 1$. Let

$$A_{01} = \{i \leq kl : s_i \notin Y \quad \text{and} \quad s_i^{\complement} \in Y\}$$
$$A_{10} = \{i \leq kl : s_i \in Y \quad \text{and} \quad s_i^{\complement} \notin Y\}$$
$$A_{00} = \{i \leq kl : s_i, s_i^{\complement} \notin Y\}.$$

Then $|A_{01}\cup A_{10}\cup A_{00}|\geq kl-\lfloor|Y|/2\rfloor\geq kl-(k-1)l=l$, and we can extend Y (this will clearly not decrease $\varphi'(\mathcal{C}_Y)$) to assure that $|A_{01}\cup A_{10}\cup A_{00}|=l$. Now define $X\subseteq\{0,1\}^{[m]}$ by

$$X=(\langle 1\rangle\cap Y)\cup\{s_i^{\complement}:i\in A_{10}\}\cup\{s_i,s_i^{\complement}:i\in A_{00}\}.$$

It is easy to check that $\{i\leq kl:X\in C_i\}=A_{01}\cup A_{10}\cup A_{00}$, and therefore $X\in\mathcal{U}$. We also have $X\cap Y=\langle 1\rangle$, hence $X\notin\mathcal{C}_Y$, and the set $\mathcal{U}\setminus\mathcal{C}_Y$ is nonempty. This shows that $\mathcal{U}$ is not included in any set of submeasure <1, and therefore that $\varphi'(\mathcal{U})=1$. ∎

Claim 2. If $\nu\leq\varphi'$ is a measure, then $\nu(\mathcal{U})\leq\frac{1}{(k-1)l^2+l}$.

Proof. Note that $\varphi'(C_i\cap\mathcal{U})\leq\varphi(\mathcal{C}_{\{s_i,s_i^{\complement}\}})/(2(k-1)l+2)=1/((k-1)l+1)$ for every i. (It is easy to see that we do have the equality here, but we shall not need this fact.) Also, every $X\in\mathcal{U}$ belongs to l many C_i. Therefore we have

$$\nu(\mathcal{U})\leq\frac{1}{n}\sum_{i=1}^{kl}\nu(C_i\cap\mathcal{U})\leq\frac{1}{(k-1)l^2+l},$$

completing the proof. ∎

Since $P(\varphi)\geq P(\varphi')$, by Claims 1 and 2 we have $P(\varphi)\geq(k-1)l^2+l$ for all pairs of integers k,l such that $kl\leq 2^{m-1}$. Letting $k=2$ and $l=2^{m-2}$ we get the desired inequality, $P(\varphi)>2^{2m-5}$. ∎

The proof of Theorem 8 relates φ_m to a frequently rediscovered example of a pathological submeasure (see [16, page 163], [12], [7, page 18-05], [9, Lemma 1.8]). Let $[kl]^l$ be the family of all l-element subsets of $\{1,\ldots,kl\}$ and let $B_i=\{s\in[kl]^l: i\in s\}$ for $i\leq kl$. Define a submeasure τ_{kl} on $[kl]^l$ by

$$\tau_{kl}(\mathcal{A})=\frac{\min\{|I|:\bigcup_{i\in I}B_i\supseteq\mathcal{A}\}}{k(l-1)+1}.$$

Theorem 9. *If $m\geq kl$ then there are a subset $\mathcal{U}$ of the support N of φ_m, a mapping $f:\mathcal{U}\to[kl]^l$, and a constant D such that $\varphi_m(f^{-1}(\mathcal{A}))=D\cdot\tau_{kl}(\mathcal{A})$ for all $\mathcal{A}\subseteq[kl]^l$.*

Proof. Following the proof of Theorem 8, we fix a $1-1$ sequence $s_1,\ldots,s_{kl}$ of elements from $\{s\in\{0,1\}^{[m]}:1\in s\}$. Define $C_i=\mathcal{C}_{\{s_i,s_i^{\complement}\}}$ and $\mathcal{U}=\{X\in N:|\{i:X\in C_i\}|=l\}$, like before. For $s\in[kl]^l$ let

$$U_s=\bigcap_{i\in s}C_i\cap\mathcal{U}.$$

Then $U_s \cap U_t \cap \mathcal{U} = \emptyset$ whenever $s \neq t$. Note also that $B_i = \{s \in [kl]^l : U_s \subseteq C_i\}$ for $i \leq kl$. The proof of Theorem 8 (more precisely, of Claim 1) shows that

$$\tau_{kl}(\mathcal{A}) = \frac{\varphi_m(\bigcup_{s\in\mathcal{A}} U_s)}{2k(l-1)+2}$$

for all $\mathcal{A} \subseteq [kl]^l$, and therefore the mapping f which collapses U_s to s and $D = 2k(l-1)+2$ are as required. ∎

The major difference between our φ_m and τ_{kl} seems to be in the fact that, while τ_{kl} is highly symmetric, φ_m is not. Roughly speaking, how pathological φ_m is in some point X of its support N depends on how far X is from being an ultrafilter in $\{0,1\}^{[m]}$.

3. Remarks and questions

Although we are concerned with the qualitative rather than the quantitative results, it would be interesting to know the true value of the constant K_M. The following example gives a lower bound 2 for K_M.

Example 10. A 1-approximate homomorphism with respect to a measure which can not be α-approximated by a homomorphism for any $\alpha < 2$. Let $m = \{1,2,3\}$ and $n = \{1,2\}$ and define $H\colon \{0,1\}^{[m]} \to \{0,1\}^{[n]}$ as follows:

$$H(s) = \begin{cases} \emptyset, & \text{if } |s| \leq 1 \\ \{1\}, & \text{if } |s| = 2 \\ \{1,2\}, & \text{if } |s| = 3. \end{cases}$$

Then it is easy to check that H is a 1-approximate homomorphism with respect to the counting measure μ on $\{0,1\}^{[n]}$ and that if $\Phi\colon \{0,1\}^{[m]} \to \{0,1\}^{[n]}$ is a homomorphism, then $\mu(\Phi(s) \Delta H(s)) \geq 2$ for some s.

By Theorems 5 and 7 the invariant K_φ is, in some sense, a measure of the pathologicity of a submeasure φ. A natural question is whether K_φ is large for every sufficiently pathological submeasure φ? Of course, we first have to define precisely what we mean by "sufficiently pathological submeasure." For example, taking $P(\varphi)$ to be the "index of pathologicity" is not the right choice in this context, because there are submeasures φ with $P(\varphi)$ arbitrarily large and K_φ close to K_M. (Take φ to be equal to a measure everywhere except on a set of a small submeasure.) The following seems to be a better notion. For a submeasure φ let (recall that $\hat{\varphi}$ is the maximal nonpathological submeasure dominated by φ):

$$C(\varphi) = \frac{\|\varphi\| - \|\hat{\varphi}\|}{\|\varphi\|}.$$

Then $0 \leq C(\varphi) \leq 1$.

Question 11. Is $\lim_{t\to 1^-} \inf_{C(\varphi)=t} K_\varphi = \infty$?

By a result of Talagrand ([12]) the submeasures τ_{kl} appearing in Theorem 9 above are, in some sense, universal among the pathological submeasures. Therefore the following is a natural special case of Question 11.

Question 12. Is $\sup_{k,l} K_{\tau_{kl}} = \infty$?

Let us end this note with a restricted version of Question 3. A positive answer to this question would have some consequences related to our original motivation for studying ε-approximate homomorphisms (see [15], [4; §9], or [3, Question 8.1]).

Definition 13. Assume $\{1,\dots,n\}$ is equipped with a submeasure φ and $\varepsilon > 0$. A mapping $H\colon \{0,1\}^{[m]} \to \{0,1\}^{[n]}$ is an *ε-approximate epimorphism* (with respect to φ) if it is an ε-approximate homomorphism and for every $s \in \{0,1\}^{[n]}$ there is $t \in \{0,1\}^{[m]}$ such that $\varphi(H(t)\Delta s) \le \varepsilon$.

Question 14. Is there a universal constant K such that every ε-approximate epimorphism can be $K\cdot\varepsilon$-approximated by a homomorphism?

References

[1] J. P. R. Christensen: Some results with relation to the control measure problem, *Vector measures and applications,* Springer Lecture Notes, Vol. 644, 1978, 125–158.

[2] J. P. R. Christensen and W. Herer: On the existence of pathological submeasures and the construction of exotic topological groups, *Math. Ann.,* **213** (1975), 203–210.

[3] I. Farah: Completely Additive Liftings, *Bull. Symb. Logic,* **4** (1998), 37–54.

[4] I. Farah: Analytic ideals and their quotients, Ph.D. Thesis, University of Toronto, 1997.

[5] D. H. Fremlin: Measure algebras, in: *Handbook of Boolean algebras* (D. Monk and R. Bonnett eds.), Elsevier, 1989, 877–980.

[6] D. H. Hyers: On the stability of the linear functional equation, *Proc. Nat. Acad. Sci. U.S.A.,* **27** (1941), 222–224.

[7] N. J. Kalton: The Maharam problem, *Séminaire Initiation à l'Analyse,* **18** (1988/89), 1–13.

[8] N. J. Kalton and J. W. Roberts: Uniformly exhaustive submeasures and nearly additive set functions, *Trans. AMS,* **278** (1983), 803–816.

[9] K. Mazur: F_σ-ideals and $\omega_1\omega_1^*$-gaps in the Boolean algebra $\mathcal{P}(\omega)/I$, *Fund. Math.,* **138** (1991), 103–111.

[10] N. Pippenger: Superconcentrators, *SIAM J. Comput.,* **6** (1977), 298–304.

[11] V. A. Popov: Additive and subadditive functions on Boolean algebras, *Siberian Math. J.,* **17** (1976), 331–339.

[12] M. Talagrand: A simple example of a pathological submeasure, *Math. Ann.*, **252** (1980), 97–102.

[13] M. Talagrand: How much are increasing sets positively correlated? *Combinatorica,* **16** (1996), 243–258.

[14] M. Talagrand: Isoperimetry, logarithmic Sobolev inequalities on the discrete cube, and Margulis' graph connectivity theorem, *Geometric and functional analysis,* **3** (1993), 295–314.

[15] S. Todorcevic: Gaps in analytic quotients, *Fund. Math.,* **156** (1998), 85–97.

[16] F. Topsøe: Some remarks concerning pathological submeasures, *Math. Scand.,* **38** (1976), 159–166.

[17] S. M. Ulam: *Problems in modern mathematics,* John Wiley & Sons, 1964.

[18] S. M. Ulam and D. Mauldin: Mathematical problems and games, *Advances in Appl. Math.,* **8** (1987), 281–344.

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